Notes on scaling limits in SOS models, local operators and form factors

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Abstract

Scaling limits of the SOS and RSOS models in the regime III are considered. These scaling limits are believed to be described by the sine-Gordon model and the restricted sine-Gordon models (or perturbed minimal conformal models) respectively. We study two different scaling limits and establish the correspondence of the scaling local height operators to exponential or primary fields in quantum field theory. An integral representation for form factors is obtained in this way. In the case of the sine-Gordon model this reproduces Lukyanov's well known representation. The relation between vacuum expectation values of local operators in the sine-Gordon model and perturbed minimal models is also discussed.

1. Introduction

The solid-on-solid (SOS) and restricted solid-on-solid (RSOS) models in the vicinity of the critical points are well known to be related to the perturbed conformal field theory [1–4]. The scaling behavior of the order parameters in the RSOS models was analyzed by Huse [5]. It was shown that it is possible to define the order parameters in such a way that they possess definite scaling dimensions that coincide to those of appropriate conformal fields in the respective models of the conformal field theory.

The consideration by Huse was based on the notion of the local height probabilities, which are the simplest correlation functions in the SOS and RSOS models. In the transfer matrix approach these quantities are vacuum expectation values of local height operators. These can be naturally generalized to the multi-point local height operators. The expectation values and form factors (i.e. matrix elements in the basis of eigenvalues of the transfer matrix) can be obtained by use of the free field representation [6,7].

Earlier [8] it was shown on the example of the six-vertex model that the free field representation makes it possible to obtain the scaling limit of local operators. In this note we discuss the scaling limit of the form factors of local height operators in the regime III of the SOS and RSOS models. We use another set of order parameter variables than those chosen by Huse to get rid of an ambiguous and physically doubtful procedure of summation over boundary conditions. Instead, we consider physically more transparent linear combinations of the local height operators for given boundary conditions. As a result we obtain a set of form factors for the $\Phi_{1,3}$ perturbation of the minimal conformal models, as well as reproduce the expressions for form factors of the sine-Gordon model, obtained earlier by Lukyanov [9]. We obtain form factors in the form of multiple integrals. Note that some simple expressions for two-particle form factors in the scaling RSOS models were obtained by Delfino [10].

Another result of the paper is a derivation of the vacuum expectation values of some primary operators in the minimal models of conformal field theory from those of exponential operators of the sine-Gordon model. The first conjecture on the vacuum expectation values was made in [11]. Later it was revised [12], and some vacuum-dependent factor was added. Though the derivation of this paper is based on some assumptions, it gives evidence that, up to signs, the correct one is the former conjecture of [11]. The vacuum-dependent factor must be nothing but ± 1.

The paper is a kind of reflection on different aspects of the problem. So the structure of the paper is rather loose. In Sec. 2 the basic reference information on the SOS and RSOS models, their spaces of states, vacuums, excitations, local operators and form factors is given. The scaling limits of these models

are described in Sec. 3. The scaling limit of local height probabilities is obtained, and the normalization of the corresponding operators in the field theory is discussed. Scaling of the operators in the restricted theory is considered in Sec. 4. Sec. 5 is devoted to derivation of the representation of the form factors. In the Appendix an amusing arithmetic theorem concerning enumeration of primary operators in the minimal models is proven.

2. SOS and RSOS models

Consider the SOS model on the square lattice [13]. A variable $n_i \in \mathbb{Z} + \delta$ with some fixed complex number δ is associated to each vertex i of the lattice, while the Boltzmann weights are associated to each face of the lattice. The variables n_i must satisfy the admissibility condition

$$|n_i - n_j| = 1$$

for adjacent lattice vertices i and j. The Boltzmann weights $W\begin{bmatrix} n_4 & n_3 \\ n_1 & n_2 \end{bmatrix} u$ of a face with local variables $n_1, \ldots, n_4$ surrounding the face are given by

$$W\begin{bmatrix} n & n \pm 1 \\ n \pm 1 & n \pm 2 \end{bmatrix} u = R_0(u),$$

$$W\begin{bmatrix} n & n \pm 1 \\ n \pm 1 & n \end{bmatrix} u = R_0(u)\frac{[n \pm u][1]}{[n][1 - u]},$$

$$W\begin{bmatrix} n & n \pm 1 \\ n \mp 1 & n \end{bmatrix} u = R_0(u)\frac{[n \pm 1][u]}{[n][1 - u]}$$

with square brackets defined in terms of the Jacobi theta function $\theta_1(u; \tau)$ of quasiperiods 1 and τ:

$$[u] = \sqrt{\frac{\pi}{\epsilon r}}\, e^{\frac{1}{4}\epsilon r}\, \theta_1\left(\frac{u}{r}; \frac{i\pi}{\epsilon r}\right).$$

The function $R_0(u)$ is an arbitrary normalization factor. The parameters r and ϵ label families of commuting transfer matrices, while the parameter u is the spectral parameter enumerating the commuting transfer matrices in each family. The region

$$\epsilon > 0, \qquad r \geq 1, \qquad 0 < u < 1$$

is called the regime III.

The space of states $\mathcal{H}_m^{(0)}$ of the SOS model in the regime III with the boundary condition m is spanned by the vectors $|P\rangle$, labeled by paths $P = \{n_k\}_{k=-\infty}^{\infty}$, such that $n_k \in \mathbb{Z} + \delta$ and the following conditions are satisfied:

(i) $|n_{k+1} - n_k| = 1$ (admissibility condition);

(ii) $n_{2k} = n_\infty, n_{2k+1} = n_\infty + 1$ or $n_{2k} = n_\infty + 1, n_{2k+1} = n_\infty$ for $|k| \gg 1$ (boundary condition).

(2.1)

In fact, not every value of n_∞ is admissible. The value of n_∞ must satisfy the condition $Nr < \operatorname{Re} n_\infty < (N + 1)r - 1$ for some integer N. The subscript m in the notation $\mathcal{H}_m^{(0)}$ is defined as follows:

$$m = n_\infty - N.$$

The space of states $\mathcal{H}_m^{(0)}$ admits two integrability preserving restrictions for particular values of the parameters of the model. First, for $\delta = 0$ it can be restricted to the space $\mathcal{H}_m^{(1)} \subset \mathcal{H}_m^{(0)}$ spanned by the paths $P = \{n_k\}$, such that

$$\forall k: \ n_k > 0, \qquad m > 0. \tag{2.2}$$

In addition, for rational r,

$$r = \frac{q}{q - p}, \qquad q > p > 0, \qquad q \text{ and } p \text{ being coprimes}, \tag{2.3}$$

it admits the restriction to the space $\mathcal{H}_m^{(2)} \subset \mathcal{H}_m^{(1)}$ spanned by the paths subject to

$$\forall k: \ 0 < n_k < q, \qquad 0 < m < p. \tag{2.4}$$

The models defined on the spaces $\mathcal{H}_m^{(1)}$ and $\mathcal{H}_m^{(2)}$ are called *restricted* solid-on-solid (RSOS) models. To distinguish the two situations we shall call them RSOS$^{(1)}$ and RSOS$^{(2)}$ models respectively. At the critical point the SOS model gives the free boson field theory in the scaling limit, while the RSOS$^{(1)}$ and RSOS$^{(2)}$ models give conformal models with the central charge of the Virasoro algebra

$$c = 1 - \frac{6}{r(r-1)}. \tag{2.5}$$

The RSOS$^{(2)}$ models give the minimal conformal models $M(p,q)$ with a finite number $(p-1)(q-1)/2$ of primary fields [4]. Below, we shall denote by $\mathcal{H}_m$ any of the spaces $\mathcal{H}_m^{(i)}$ $(i = 0, 1, 2)$, when we do not want to specify a restriction.

The eigenvector of the transfer matrix in the space $\mathcal{H}_m$ of the eigenvalue of the largest absolute value will be called the vacuum $|\text{vac}\rangle_m$. It is convenient to normalize the weights so that this largest eigenvalue would be equal to one, which is achieved by appropriate choice of $R_0(u)$. Other eigenvectors of the transfer matrix can be described in terms of elementary excitations (particles) $A_s(\theta)$ with the 'rapidity' θ and the 'internal state' s. The vector $|A_{s_1}(\theta_1) \ldots A_{s_N}(\theta_N)\rangle_m$ corresponds to the eigenvalue factorizing into a product $\prod_{j=1}^N \tau_{s_j}(\theta_j)$ of one-particle functions $\tau_s(\theta)$. There are two types of excitations in the SOS model: kinks and breathers. Since breathers can be considered as bound states of two kinks, we shall only describe the kinks. The one-kink function is

$$\tau_{\text{kink}}(\theta) = \tau(\theta + i\pi u), \qquad \tau(\theta) = \frac{\theta_4\left(\frac{1}{4} + \frac{\theta}{2\pi i}; \frac{i\pi}{2\epsilon}\right)}{\theta_4\left(\frac{1}{4} - \frac{\theta}{2\pi i}; \frac{i\pi}{2\epsilon}\right)}.$$

An internal state of a kink is described by a pair of admissible vacuum labels m_1, m_2, $|m_2 - m_1| = 1$, so that the multi-kink eigenvectors can be written as $|A(\theta_1)_m^{m_1} A(\theta_2)_{m_1}^{m_2} \ldots A(\theta_N)_{m_{N-1}}^m\rangle_m$.

We shall study the scaling limit $\epsilon \to 0$. In this limit

$$\tau(\theta) = 1 + iM\,\text{sh}\,\theta + O(M^2), \qquad \epsilon \to 0, \qquad -\frac{\pi^2}{2\epsilon} < \theta < \frac{\pi^2}{2\epsilon},$$

where

$$M = 4e^{-\pi^2/2\epsilon} \tag{2.6}$$

is the kink mass. Because of this particular form of the mass we shall often use the parameter $M/4$, which will be always written down explicitly to avoid a confusion.

Consider any operator O acting in the space $\mathcal{H}_m$. Its form factors are defined as

$$F(O|\theta_1, \ldots, \theta_N)_{mm_1\ldots m_{N-1}} = {}_m\langle\text{vac}|O|A(\theta_1)_m^{m_1} A(\theta_2)_{m_1}^{m_2} \ldots A(\theta_N)_{m_{N-1}}^m\rangle_m.$$

Below we need some objects defined on a half-line and related to the corner transfer matrix picture. Define the space $\mathcal{H}_{mn}^{(0)}$ of semi-infinite paths $|\{n_k\}_{k=0}^\infty\rangle$ subject to (2.1) with $n_0 = n$. The spaces $\mathcal{H}_{mn}^{(1)}$ and $\mathcal{H}_{mn}^{(2)}$ are defined by the additional conditions (2.2) and (2.4) correspondingly. Again, $\mathcal{H}_{mn}$ will denote any of these spaces. There are some operators defined on these spaces

$$H : \mathcal{H}_{mn} \to \mathcal{H}_{mn}, \qquad \Psi^*(\theta)_m^{m\pm 1} : \mathcal{H}_{mn} \to \mathcal{H}_{m\pm 1, n}$$

called corner Hamiltonian and type II vertex operators.[1] The operator $e^{-2\pi H}$ is proportional to the product of the four corner transfer matrices. The operator H possesses an equidistant spectrum spaced by $2\epsilon/\pi$. Its eigenvalues are degenerate. The multiplicities of this degeneration are given by the generating functions

$$\chi_{mn}(z) = \text{Tr}_{\mathcal{H}_{mn}}\, z^{\pi H/2\epsilon} = z^{\Delta_{mn}} \sum_{k=0}^\infty z^k \dim \mathcal{H}_{mn}(k),$$

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[1]Since we limit our consideration by the one-point local height operators, we do not need the type I vertex operators.

where Δ_{mn} is the lowest eigenvalue of the operator $\pi H/2\epsilon$ on the space $\mathcal{H}_{mn}$, and $\mathcal{H}_{mn}(k)$ is its eigen-subspace corresponding to the eigenvalue $\Delta_{mn} + k$. It turns out [1] that the minimal eigenvalues are equal to

$$\Delta_{mn} = \frac{(rm - (r-1)n)^2 - 1}{4r(r-1)}, \tag{2.7}$$

while the generating functions in the three cases are given by

$$\chi_{mn}^{(0)}(z) = \frac{z^{\Delta_{mn}}}{\prod_{k=1}^{\infty}(1 - z^k)}, \tag{2.8}$$

$$\chi_{mn}^{(1)}(z) = \chi_{mn}^{(0)}(z) - \chi_{m,-n}^{(0)}(z), \tag{2.9}$$

$$\chi_{mn}^{(2)}(z) = \sum_{k \in \mathbb{Z}}(\chi_{m,n+2qk}^{(0)}(z) - \chi_{m,-n+2qk}^{(0)}(z)). \tag{2.10}$$

Note, that for m and n being integers this formula gives highest weights of the degenerate representations of the Virasoro algebra with the central charge given by (2.5). The characters $\chi_{mn}^{(1)}(z)$ and $\chi_{mn}^{(2)}(z)$ coincide with the characters of the respective irreducible representations of the Virasoro algebra.

For future use we introduce the notation

$$\alpha_+ = \sqrt{2\frac{r}{r-1}}, \qquad \alpha_- = -\sqrt{2\frac{r-1}{r}}, \qquad \alpha_0 = \frac{\alpha_+ + \alpha_-}{2} = \frac{1}{\sqrt{2r(r-1)}} \tag{2.11}$$

and

$$\alpha_{mn} = \frac{1-m}{2}\alpha_+ + \frac{1-n}{2}\alpha_-. \tag{2.12}$$

With this notation we have

$$c = 1 - 12\alpha_0^2, \qquad \Delta_{mn} = \frac{(\alpha_{mn} - \alpha_0)^2 - \alpha_0^2}{2}. \tag{2.13}$$

It turns out that for $m, n \in \mathbb{Z}_{>0}$ the space $\mathcal{H}_{m,-n}^{(0)}$ can be immersed into $\mathcal{H}_{mn}^{(0)}$ as a subspace, so that $\mathcal{H}_{mn}^{(1)} \simeq \mathcal{H}_{mn}^{(0)}/\mathcal{H}_{m,-n}^{(0)}$. Similarly, $\mathcal{H}_{mn}^{(2)} \simeq \mathcal{H}_{mn}^{(0)}/(\mathcal{H}_{m,-n}^{(0)} \cup \mathcal{H}_{m,2q-n}^{(0)})$. Then the formulas (2.9) and (2.10) are generalized to the identities

$$\mathrm{Tr}_{\mathcal{H}_{mn}^{(1)}} X = \mathrm{Tr}_{\mathcal{H}_{mn}^{(0)}} X - \mathrm{Tr}_{\mathcal{H}_{m,-n}^{(0)}} X, \tag{2.14}$$

$$\mathrm{Tr}_{\mathcal{H}_{mn}^{(2)}} X = \sum_{k \in \mathbb{Z}}(\mathrm{Tr}_{\mathcal{H}_{m,n+2qk}^{(0)}} X - \mathrm{Tr}_{\mathcal{H}_{m,-n+2qk}^{(0)}} X) \tag{2.15}$$

for any operator X on the space $\mathcal{H}_{mn}^{(i)}$ realized as a factor space.

The quantities

$$\chi_{mn} = \chi_{mn}(x^4) = \mathrm{Tr}_{\mathcal{H}_{mn}} e^{-2\pi H}, \qquad x = e^{-\epsilon},$$

are of physical importance: they determine the local height probabilities. Namely, let Π_{mn} be the projector in the space $\mathcal{H}_m$ onto the subspace spanned by the vectors $|\{n_k\}\rangle$ with $n_0 = n$. The local height probabilities are

$$P_{mn} = \langle \Pi_{mn} \rangle = \frac{[n]\chi_{mn}}{\sum_n [n]\chi_{mn}}.$$

The sum in the denominator is taken over all admissible values of n, namely, $n \in \mathbb{Z} + \delta$ in the case of the SOS model and over $n \in \mathbb{Z}_{>0}$ and $n = 1, 2, \ldots, q-1$ in the cases RSOS$^{(1)}$ and RSOS$^{(2)}$ correspondingly.

The corner Hamiltonian and vertex operators satisfy the relations

$$[H, \Psi^*(\theta)_m^{m'}] = i\frac{d}{d\theta}\Psi^*(\theta)_m^{m'}, \tag{2.16}$$

$$\Psi^*(\theta_1)_s^{m'}\Psi^*(\theta_2)_m^s = \sum_{s'} S\begin{bmatrix} m' & s' \\ s & m \end{bmatrix} \theta_1 - \theta_2 \end{bmatrix} \Psi^*(\theta_2)_{s'}^{m'}\Psi^*(\theta_1)_m^{s'}, \tag{2.17}$$

$$\Psi^*(\theta')_{m''}^{m'}\Psi^*(\theta)_m^{m''} = -\frac{i[m'']'}{\theta' - \theta - i\pi}\delta_{m'm} + O(1) \text{ as } \theta' \to \theta + i\pi. \tag{2.18}$$

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Here $[u]' = [u]|_{r \to r-1}$, and the kink S matrix

$$S\begin{bmatrix} m_4 & m_3 \\ m_1 & m_2 \end{bmatrix}\theta\Bigg] = - \, W\begin{bmatrix} m_4 & m_1 \\ m_3 & m_2 \end{bmatrix}\frac{\theta}{i\pi}\Bigg]\Bigg|_{r \to r-1} \tag{2.19}$$

is expressed in terms of the Boltzmann weights with the normalization factor $R_0(u)$ chosen so that the largest eigenvalue is equal to one for $0 < u < 1$, and with the shifted parameter r. The products of type II vertex operators on the half line represent many-kink states in the space of states $\mathcal{H}_m$ on the full line.

The form factors of the local height operator Π_{mn} are known to be given by the trace function

$$F(\Pi_{mn}|\theta_1, \ldots, \theta_N)_{mm_1 \ldots m_{N-1}} = \frac{[n]}{[m]'\chi} \, \mathrm{Tr}_{\mathcal{H}_{mn}}(e^{-2\pi H}\Psi^*(\theta_N)_{m_{N-1}}^m \ldots \Psi^*(\theta_1)_m^{m_1}) \tag{2.20}$$

with

$$[m]'\chi = \sum_n [n]\chi_{mn}. \tag{2.21}$$

The value of χ is the same for all three cases of SOS, RSOS$^{(1)}$ and RSOS$^{(2)}$ models:

$$\chi = \frac{2x^{-1/r(r-1)}}{\prod_{k=0}^{\infty}(1 - x^{2+4k})}.$$

3. Scaling limits in the unrestricted case: the problem of identification and normalization of fields

The unrestricted SOS model admits two different types of scaling limits:

$$\mathrm{I}: \quad \epsilon \to 0, \qquad \delta = \mathrm{const}, \tag{3.1}$$

$$\mathrm{II}_{\pm}: \quad \epsilon \to 0, \qquad \delta = \pm\frac{i\pi}{2r\epsilon}. \tag{3.2}$$

Consider the first scaling limit (3.1), which is consistent with the restrictions for $\delta = 0$. In this limit the S matrix (2.19) tends to the following one

$$S\begin{bmatrix} m & m\pm 1 \\ m\pm 1 & m\pm 2 \end{bmatrix}\theta\Bigg] = S_0(\theta),$$

$$S\begin{bmatrix} m & m\pm 1 \\ m\pm 1 & m \end{bmatrix}\theta\Bigg] = S_0(\theta)\frac{\mathrm{sh}\,\frac{i\pi m\pm\theta}{r-1}\,\mathrm{sh}\,\frac{i\pi}{r-1}}{\mathrm{sh}\,\frac{i\pi m}{r-1}\,\mathrm{sh}\,\frac{i\pi-\theta}{r-1}}, \tag{3.3}$$

$$S\begin{bmatrix} m & m\pm 1 \\ m\mp 1 & m \end{bmatrix}\theta\Bigg] = S_0(\theta)\frac{\mathrm{sh}\,\frac{i\pi(m\mp 1)}{r-1}\,\mathrm{sh}\,\frac{\theta}{r-1}}{\mathrm{sh}\,\frac{i\pi m}{r-1}\,\mathrm{sh}\,\frac{i\pi-\theta}{r-1}},$$

where

$$S_0(\theta) = -\exp\left(2i\int_0^{\infty} \frac{dt}{t} \, \frac{\mathrm{sh}\,\frac{\pi t}{2}\,\mathrm{sh}\,\frac{\pi(r-2)t}{2}\sin\theta t}{\mathrm{sh}\,\pi t\,\mathrm{sh}\,\frac{\pi(r-1)t}{2}}\right). \tag{3.4}$$

Consider now one of the scaling limits (3.2), e. g., the II_- limit. Because of a nonzero value of δ this scaling limit is inconsistent with the restrictions. The S matrix (2.19) tends, up to a simple factor, to the kink S matrix $S_{SG;\beta}(\theta)$ of the sine-Gordon model with the action

$$\mathcal{A} = \int d^2x \left(\frac{(\partial_\mu\varphi)^2}{8\pi} + \mu\cos\beta\varphi\right), \qquad \beta = \sqrt{2\frac{r-1}{r}} = -\alpha_-. \tag{3.5}$$

Namely,

$$S\begin{bmatrix} m_4 & m_3 \\ m_1 & m_2 \end{bmatrix}\theta\Bigg] \to e^{\theta\frac{\varepsilon_1-\varepsilon_1'}{2(r-1)}}\,e^{i\pi\frac{\varepsilon_1-\varepsilon_2'}{2(r-1)}}\,S_{SG;\beta}(\theta)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'} \tag{3.6}$$

with $\varepsilon_1 = m_1 - m_2$, $\varepsilon_2 = m_4 - m_1$, $\varepsilon'_1 = m_4 - m_3$, $\varepsilon'_2 = m_3 - m_2$. The parameter μ in the action is related to the kink mass [15] as

$$\mu = (M/4)^{2/r}\check{\mu}, \qquad \check{\mu} = \frac{2}{\pi}\frac{\Gamma(1-1/r)}{\Gamma(1/r)}\left(\frac{2\sqrt{\pi}\Gamma(r/2)}{\Gamma((r-1)/2)}\right)^{2/r}. \tag{3.7}$$

The S matrix (3.3), (3.4) of the scaling SOS model with finite δ is related to the S matrix of the sine-Gordon model $S_{SG;\beta}(\theta)$ by means of the vertex-face correspondence, basically described in [14]. From the first glance, it seems that both describe the same model in different bases of elementary excitation, but it is not quite correct. To understand it better, let us recall the vertex-face correspondence between the lattice SOS model and the eight-vertex model [13]. The eight-vertex model corresponds to the compactification of the sine-Gordon model, where $\varphi \sim \varphi + 4\pi/\beta$ so that the fields $e^{il\beta\varphi/2}$ with $l \in \mathbb{Z}$ are the only local neutral exponential operators. The vacuum of this model is double degenerated while the degeneracy of the SOS model is infinite. On the other hand, every eigenvector of the transfer matrix of the SOS model is mapped onto an eigenvector in the eight-vertex model with the same eigenvalue. It means that the SOS model is an extension of the eight-vertex model, where each vector of the eight-vertex model is infinitely repeated in the SOS model. Hence, the scaling SOS model is an extension of the compactified sine-Gordon model. More precisely, we have a one-parametric family (parameterized by $\delta \in \mathbb{C}/\mathbb{Z}$) of extensions of the compactified sine-Gordon model. For $\delta = 0$ the extension admits one or two restrictions described above. In the limits $\delta \to \pm\infty$ the extention coincides with the usual non-compact sine-Gordon model. The most important fact, which will be used below, is that the vacuums $|vac\rangle_m$ for finite δ are not linear combinations of the vacuums of the full sine-Gordon theory, since they are living, strictly speaking, in another theory. In particular, the restricted sine-Gordon model is the restriction just of the $\delta = 0$ scaling SOS model rather than that of the genuine sine-Gordon model.

Let us start our analysis from the unrestricted SOS model. Instead of local height operators themselves it is more convenient to consider their Fourier transforms:

$$\Pi_m(a) = \sum_{n \in \mathbb{Z}+m} \frac{e^{i\pi\frac{an}{r}}}{\sin\frac{\pi n}{r}}\Pi_{mn}.$$

The denominator $\sin\frac{\pi n}{r}$ is introduced to get rid (in the scaling limit) of the n dependence related to the factor $[n]$ before the trace in the form factor (2.20).

First, consider the vacuum expectation value of the operator $\Pi_m(a)$:

$$P_m(a) = \langle\Pi_m(a)\rangle = \sum_{n \in \mathbb{Z}+m} \frac{[n]\chi_{mn}^{(0)}}{[m]'\chi_m^{(0)}}\frac{e^{i\pi\frac{an}{r}}}{\sin\frac{\pi n}{r}}.$$

Let us calculate $P_m(a)$ in the first scaling limit (3.1). In this limit

$$[u] \simeq \sqrt{\frac{\pi}{\epsilon r}}\,e^{-\pi^2/4\epsilon r}\sin\frac{\pi u}{r}, \qquad \prod_{n=0}^{\infty}\frac{1-x^{2+4n}}{1-x^{4+4n}} \simeq 2\sqrt{\frac{\epsilon}{\pi}}.$$

Substituting $n = m + k$ we obtain

$$P_m(a) \simeq \sqrt{\frac{\epsilon}{\pi}\frac{r-1}{r}}\frac{\exp\left(i\pi\frac{am}{r} + \frac{\pi^2}{4\epsilon r(r-1)} - \frac{\epsilon m^2}{r(r-1)}\right)}{\sin\frac{\pi m}{r-1}}\sum_{k\in\mathbb{Z}}e^{-\epsilon\frac{r-1}{r}k^2 + 2\epsilon\frac{mk}{r} + i\pi\frac{ak}{r}}$$

The sum in the r. h. s. is a theta function, which admits the modular transformation:

$$\sum_{k\in\mathbb{Z}}e^{i\pi\tau k^2 + 2\pi iku} = (-i\tau)^{-1/2}\sum_{k\in\mathbb{Z}}e^{-i\pi(u+k)^2/\tau}.$$

Here $u = \frac{a}{2r} - \frac{i\epsilon m}{\pi r}$, $\tau = \frac{i\epsilon}{\pi}\frac{r-1}{r}$. In the limit $\tau \to 0$ for $-k - \frac{1}{2} < \operatorname{Re} u < -k + \frac{1}{2}$ the leading term in the r. h. s. is $e^{-i\pi(u+k)^2/\tau}$. It means that for

$$-r < \operatorname{Re} a + \frac{2\epsilon}{\pi}\operatorname{Im} m < r \tag{3.8}$$

the only term with $k = 0$ survives. For real m and a the validity region is

$$-r < a < r. \tag{3.9}$$

Finally, we have

$$P_m(a) \simeq \frac{e^{i\pi \frac{am}{r-1}}}{\sin \frac{\pi m}{r-1}} \left(\frac{M}{4}\right)^{\frac{a^2-1}{2r(r-1)}}, \tag{3.10}$$

where M is the kink mass (2.6). The expectation value of a local operator must be proportional to $M^{2\Delta}$ with Δ being the conformal dimension of the operator. We conclude that the conformal dimension of the operator $\Pi_m(a)$ is given by

$$\Delta(a) = \frac{a^2 - 1}{4r(r-1)}.$$

As the minimal dimension is equal to $-1/4r(r-1)$ we can think that this model must be identified with a perturbation of the twisted free boson with the energy-momentum tensor

$$T(z) = -\frac{1}{2}(\partial\varphi)^2 + i\alpha_0\partial^2\varphi.$$

The central charge of the corresponding Virasoro algebra $c = 1 - 12\alpha_0^2$ coincides with (2.5). The dimension of the exponential operator $e^{i\alpha\varphi(x)}$ is equal to

$$\Delta_\alpha = \frac{(\alpha - \alpha_0)^2 - \alpha_0^2}{2}. \tag{3.11}$$

In particular, for $\alpha = \alpha_{mn}$ this expression gives the conformal weights Δ_{mn}.

Let us write the scaling limit of the operator $\Pi_m(a)$ as a value of some local field $\Phi_\alpha(x)$ of the dimension Δ_α:

$$\Pi_m(a) \simeq \Phi_\alpha(0), \qquad a = \frac{\alpha_0 - \alpha}{\alpha_0}. \tag{3.12}$$

For small enough $|\alpha - \alpha_0|$ there are two such fields in the theory: $e^{i\alpha\varphi(x)}$ and $e^{i(2\alpha_0-\alpha)\varphi(x)}$. Hence, the most general form of the operator $\Phi_\alpha(x)$ is

$$\Phi_\alpha(x) = f_{\alpha,m}e^{i\alpha\varphi(x)} + g_{\alpha,m}e^{i(2\alpha_0-\alpha)\varphi(x)}. \tag{3.13}$$

Equivalently, we can write

$$e^{i\alpha\varphi(x)} = \frac{1}{2i}(A_{\alpha,m}\Phi_\alpha(x) + B_{\alpha,m}\Phi_{2\alpha_0-\alpha}(x)). \tag{3.14}$$

The coefficients $f_{\alpha,m}$, $g_{\alpha,m}$ and $A_{\alpha,m}$, $B_{\alpha,m}$ may depend on the vacuum m. From the fact that $\Phi_0(x) - \Phi_{2\alpha_0}(x) = 2i$ we get

$$A_{0,m} = -B_{0,m} = 1. \tag{3.15}$$

We shall fix the coefficients $A_{\alpha,m}$, $B_{\alpha,m}$ later.

Now we discuss the normalization of the operators $\Phi_\alpha(x)$. We shall need this later to compare our results to the exact vacuum expectation values [11, 12]. The normalization of local operators is usually fixed by the short range (ultraviolet) asymptotics of their pair correlation functions. Note that the next consideration concerns an *arbitrary* field of the form (3.13), since it refers no other properties of the particular field (3.12).

Consider first the operator product expansions of the exponential operators. Let α_1, α_2 be arbitrary real numbers. Let $\alpha_{12}(k,l) = \alpha_1 + \alpha_2 - 2\alpha_{kl}$. Let Δ_1, Δ_2, and $\Delta_{12}(k,l)$ be the corresponding conformal dimensions according to the formula (3.11). From conformal perturbation theory we know the following general form of the operator product expansion of two exponential operators:

$$e^{i\alpha_1\varphi(x)}e^{i\alpha_2\varphi(y)} = \sum_{k,l\in\mathbb{Z}} |x-y|^{2\Delta_{12}(k,l)-2\Delta_1-2\Delta_2} D^{kl,m}_{\alpha_1,\alpha_2}(M|x-y|)e^{i\alpha_{12}(k,l)\varphi(y)}$$

$$+ \text{(contributions of descendants)}$$

with some structure functions $D^{kl,m}_{\alpha_1,\alpha_2}(z)$. The values $D^{kl,m}_{\alpha_1,\alpha_2}(0)$ are expressed in terms of the multiple integrals over the Euclidean plane defined in [17, 18]. Some of these coefficients are very simple. In particular, we know that $D^{11,m}_{\alpha_1,\alpha_2}(0) = 1$. The other coefficients are rather complicated. Fortunately, some information about these constants is encoded in the vacuum expectation values of the exponential operators.

To fix normalization of the operators Φ_α let us consider the ultraviolet limit of the correlation functions $\langle e^{i\alpha\varphi(x)} e^{i\alpha\varphi(y)} \rangle_m$, $\langle e^{i\alpha\varphi(x)} e^{i(2\alpha_0-\alpha)\varphi(y)} \rangle_m$. We are interesting in the contributions proportional to $|x - y|^{-4\Delta_\alpha}$. For the latter of the two functions we easily get

$$\langle e^{i\alpha\varphi(x)} e^{i(2\alpha_0-\alpha)\varphi(y)} \rangle_m = |x - y|^{-4\Delta_\alpha} \langle e^{i2\alpha_0\varphi(y)} \rangle_m + \ldots$$

The dots designate all other contributions into the short range asymptotics. Indeed, there are two fields of the dimension 0, the unit operator and $e^{i2\alpha_0\varphi}$, that can appear in the expansion of the operator product in the l. h. s. But the coefficient at the unit operator $D^{-1-1,m}_{\alpha,2\alpha_0-\alpha}(0)$ is known to be zero for any value of α. Then the r. h. s. is determined by the second operator.

The product $\langle e^{i\alpha\varphi(x)} e^{i\alpha\varphi(y)} \rangle_m$ is more complicated. Its part proportional to $|x - y|^{4\Delta_\alpha}$ depends on the parameter α discontinuously. For generic α the respective part vanishes. But for the discrete values $\alpha = \alpha_{kl}$ $(k, l \in \mathbb{Z})$ we have

$$\langle e^{i\alpha\varphi(x)} e^{i\alpha\varphi(y)} \rangle_m = |x - y|^{-4\Delta_\alpha} D^{kl,m}_{\alpha,\alpha}(0) + \ldots$$

Here, on the contrary, the only nonzero contribution is that of the unit operator, while the contribution of the operator $e^{i2\alpha_0\varphi}$ vanishes: $D^{k+2\,l+2,m}_{\alpha,\alpha}(0) = 0$.

The value $D^{kl,m}_{\alpha,\alpha}(0)$ can be easily evaluated using the 'continuation' from the sinh-Gordon theory in the spirit of [11, 12]. In the sinh-Gordon theory, where all αs and β are imaginary and the vacuum is unique, we get

$$\langle e^{i\alpha\varphi(x)} e^{i\alpha\varphi(y)} \rangle = \langle e^{i\alpha\varphi(x)} e^{i(2\alpha_0-\alpha)\varphi(x)} \rangle \frac{\langle e^{i\alpha\varphi} \rangle}{\langle e^{i(2\alpha_0-\alpha)\varphi} \rangle} + \ldots = |x - y|^{-4\Delta_\alpha} \frac{\langle e^{i\alpha\varphi} \rangle \langle e^{i2\alpha_0\varphi} \rangle}{\langle e^{i(2\alpha_0-\alpha)\varphi} \rangle} + \ldots.$$

Though the first and second equalities taken separately are, of course, wrong in the sine-Gordon theory, the final result must be valid for the special fields:

$$\langle e^{i\alpha\varphi(x)} e^{i\alpha\varphi(y)} \rangle_m = |x - y|^{-4\Delta_\alpha} \frac{\langle e^{i\alpha\varphi} \rangle_m \langle e^{i2\alpha_0\varphi} \rangle_m}{\langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m} + \ldots \quad \text{for } \alpha = \alpha_{kl}. \tag{3.16}$$

Note that, despite of its form in terms of vacuum expectation values of some massive theory, the coefficient in the r. h. s. is nothing but a structure constant of a kind of conformal field theory, and the 'analytic continuation' is simply a roundabout way to express the result of the integrations in a convenient form.

For the operators $\Phi_\alpha(x)$ of the form (3.13) we obtain

$$\langle \Phi_\alpha(x) \Phi_\alpha(y) \rangle = |x - y|^{-4\Delta_\alpha} c_{\alpha,m} + \ldots,$$
$$\langle \Phi_\alpha(x) \Phi_{2\alpha_0-\alpha}(y) \rangle = |x - y|^{-4\Delta_\alpha} c'_{\alpha,m} + \ldots,$$

where

$$c_{\alpha,m} = \frac{\langle e^{i2\alpha_0\varphi} \rangle_m}{\langle e^{i\alpha\varphi} \rangle_m \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m} \times \begin{cases} 2 f_{\alpha,m} g_{\alpha,m} \langle e^{i\alpha\varphi} \rangle_m \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m, & \alpha \notin \{\alpha_{kl}\}, \\ \left(f_{\alpha,m} \langle e^{i\alpha\varphi} \rangle_m + g_{\alpha,m} \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m \right)^2, & \alpha \in \{\alpha_{kl}\}, \end{cases} \tag{3.17}$$

and

$$c'_{\alpha,m} = \frac{\langle e^{i2\alpha_0\varphi} \rangle_m}{\langle e^{i\alpha\varphi} \rangle_m \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m} \times \begin{cases} (f_{\alpha,m} f_{2\alpha_0-\alpha,m} + g_{\alpha,m} g_{2\alpha_0-\alpha,m}) \langle e^{i\alpha\varphi} \rangle_m \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m, & \alpha \notin \{\alpha_{kl}\}, \\ \left(f_{\alpha,m} \langle e^{i\alpha\varphi} \rangle_m + g_{\alpha,m} \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m \right) \times \\ \quad \left(f_{2\alpha_0-\alpha,m} \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m + g_{2\alpha_0-\alpha,m} \langle e^{i\alpha\varphi} \rangle_m \right), & \alpha \in \{\alpha_{kl}\}. \end{cases} \tag{3.18}$$

For generic values of α these expressions are rather ugly, while for special ones they take a nice form:

$$\begin{aligned} c_{\alpha,m} &= \frac{\langle e^{i2\alpha_0\varphi} \rangle_m}{\langle e^{i\alpha\varphi} \rangle_m \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m} \langle \Phi_\alpha \rangle^2_m, \\ c'_{\alpha,m} &= \frac{\langle e^{i2\alpha_0\varphi} \rangle_m}{\langle e^{i\alpha\varphi} \rangle_m \langle e^{i(2\alpha_0-\alpha)\varphi} \rangle_m} \langle \Phi_\alpha \rangle_m \langle \Phi_{2\alpha_0-\alpha} \rangle_m \quad \text{for } \alpha = \alpha_{kl}. \end{aligned} \tag{3.19}$$

Consider the scaling limit II$_-$. For the expectation values we have

$$P_m(a) \simeq 2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\frac{(a-1)m}{r-1}}\,\mathrm{e}^{-\frac{\pi^2(a^2-1)}{4r(r-1)}} = 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi\frac{(a-1)s}{r-1}}\left(\frac{M}{4}\right)^{(a-1)^2/2r(r-1)} = 2\mathrm{i}\mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s}\left(\frac{M}{4}\right)^{\alpha^2}, \qquad (3.20)$$

where

$$s = \delta - m$$

is a finite integer, which labels the degenerate sine-Gordon vacuums. Denote these vacuums as $|\mathrm{vac}\rangle_s$.

It means that the conformal dimension of the field corresponding to the leading contribution to the operator $\Pi_m(a)$ in this limit

$$\Delta'_\alpha = \frac{\alpha^2}{2}$$

coincides with that of the exponential field $\mathrm{e}^{\mathrm{i}\alpha\varphi}$ in the free field theory with the energy-momentum tensor $T(z) = -\frac{1}{2}(\partial\varphi)^2$.

In this limit, we can identify the operator $\Pi_m(a)$ with the exponential operator $\mathrm{e}^{\mathrm{i}\alpha\varphi}$. The factor $\mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s}$ in (3.20) is interpreted in terms of multiple vacuums in the sine-Gordon model. Namely, the vacuum $|\mathrm{vac}\rangle_s$ is defined by the expectation value of the field $\varphi(x)$:

$$\langle\varphi(x)\rangle_s \equiv {}_s\langle\mathrm{vac}|\varphi(x)|\mathrm{vac}\rangle_s = \frac{2\pi s}{\beta}, \qquad s \in \mathbb{Z}. \qquad (3.21)$$

Let us fix the normalization of the exponential operators by the ultraviolet limit of their pair correlation functions:

$$\langle\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}\mathrm{e}^{-\mathrm{i}\alpha\varphi(y)}\rangle_s \simeq |x - y|^{-2\alpha^2} \quad \text{as} \quad |x - y| \to 0. \qquad (3.22)$$

With this normalization condition, the vacuum expectation values of the exponential operators are known to be [11]

$$\langle\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}\rangle_s = G_\alpha \mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s} \equiv (M/4)^{\alpha^2}\check{G}_\alpha \mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s}, \quad \text{where}$$

$$\check{G}_\alpha = \left(2\sqrt{\pi}\frac{\Gamma(r/2)}{\Gamma((r-1)/2)}\right)^{\alpha^2}\exp\int_0^\infty \frac{dt}{t}\left(\frac{\mathrm{sh}^2\,\alpha\beta t}{2\,\mathrm{sh}\frac{\beta^2 t}{2}\,\mathrm{ch}\frac{(2-\beta^2)t}{2}\,\mathrm{sh}\,t} - \alpha^2 \mathrm{e}^{-2t}\right). \qquad (3.23)$$

The phase factor $\mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s}$ omitted in [11] is an immediate consequence of (3.21). Therefore, in the limit II$_-$ we have

$$\Phi_\alpha(x) = 2\mathrm{i}\check{G}_\alpha^{-1}\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}. \qquad (3.24)$$

Our next aim is to generalize this result to arbitrary values of δ. First, generalize the expression for the vacuum expectation value of the exponential field (3.23). We conjecture that

$$\langle\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}\rangle_m = (M/4)^{2\Delta_\alpha}\check{G}_\alpha \mathrm{e}^{-2\pi\mathrm{i}\frac{\alpha}{\beta}m}. \qquad (3.25)$$

All off-diagonal vacuum matrix elements of the exponential operators are supposed to be zero. Note that the last conjecture implies the fact that the vacuums $|\mathrm{vac}\rangle_m$ are not any linear combinations of the vacuums $|\mathrm{vac}\rangle_s$, otherwise some off-diagonal vacuum matrix elements would be inevitably non-zero.

Now we want to find such values of the coefficients $A_{\alpha,m}$, $B_{\alpha,m}$ that the equations (3.10), (3.15) were satisfied. Besides, we demand consistency with the relation (3.24) and a similar relation in the limit II$_+$. The solution is

$$A_{\alpha,m} = \check{G}_\alpha, \qquad B_{\alpha,m} = -\check{G}_\alpha \mathrm{e}^{-4\pi\mathrm{i}\frac{\alpha}{\beta}m}. \qquad (3.26)$$

This means that

$$\mathrm{e}^{\mathrm{i}\alpha\varphi(x)} = \frac{\check{G}_\alpha}{2\mathrm{i}}\left(\Phi_\alpha(x) - \mathrm{e}^{-4\pi\mathrm{i}\frac{\alpha}{\beta}m}\Phi_{2\alpha_0-\alpha}(x)\right), \qquad (3.27)$$

$$\Phi_\alpha(x) = \frac{\mathrm{e}^{4\pi\mathrm{i}\frac{\alpha_0}{\beta}m}}{\check{G}_\alpha \sin(4\pi\frac{\alpha_0}{\beta}m)}\left(\mathrm{e}^{\mathrm{i}\alpha\varphi(x)} + \check{R}(\alpha)\mathrm{e}^{-4\pi\mathrm{i}\frac{\alpha}{\beta}m}\mathrm{e}^{\mathrm{i}(2\alpha_0-\alpha)\varphi(x)}\right). \qquad (3.28)$$

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The function

$$\check{R}(\alpha) = \frac{\check{G}_\alpha}{\check{G}_{2\alpha_0-\alpha}} = \left(2\sqrt{\pi}\left(\frac{r}{r-1}\right)^r \frac{\Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r-1}{2}\right)}\right)^{4\alpha_0(\alpha-\alpha_0)} \frac{\Gamma(1-\alpha_+(\alpha-\alpha_0))\Gamma(1-\alpha_-(\alpha-\alpha_0))}{\Gamma(1+\alpha_+(\alpha-\alpha_0))\Gamma(1+\alpha_-(\alpha-\alpha_0))}, \quad (3.29)$$

is a part of the so called reflection function [16], which plays an important role in the Liouville field theory:

$$R(\alpha) = \frac{G_\alpha}{G_{2\alpha_0-\alpha}} = (M/4)^{4\alpha_0(\alpha-\alpha_0)}\check{R}(\alpha). \quad (3.30)$$

In the limit II$_-$ the first term in (3.28) prevails while in the limit II$_+$ it is suppressed in comparison to the second one. We have

$$\Phi_\alpha(x) = \begin{cases} 2\mathrm{i}\check{G}_\alpha^{-1}\mathrm{e}^{\mathrm{i}\alpha\varphi(x)} & \text{for } \delta = -\mathrm{i}\pi/2r\epsilon \text{ (the limit II}_-\text{)}, \\ -2\mathrm{i}\check{G}_{2\alpha_0-\alpha}^{-1}\mathrm{e}^{4\pi\mathrm{i}\frac{2\alpha_0-\alpha}{\beta}m}\mathrm{e}^{\mathrm{i}(2\alpha_0-\alpha)\varphi(x)} & \text{for } \delta = +\mathrm{i}\pi/2r\epsilon \text{ (the limit II}_+\text{)}. \end{cases}$$

This conforms (3.24) in the limit II$_-$, while in the limit II$_+$ the role of the field $\varphi(x)$ is played by the combination

$$\varphi'(x) = 4\pi m/\beta + \varphi(x), \quad (3.31)$$

which possesses finite vacuum expectation values in this limit.

Substituting the equation (3.28) into Eqs. (3.17), (3.18) we obtain the explicit form of the normalization factors:

$$c_{\alpha,m} = \frac{\check{R}(\alpha)}{\check{G}_\alpha^2\check{R}(0)} \times \begin{cases} 2\left(\frac{\mathrm{e}^{2\pi\mathrm{i}\frac{\alpha_0-\alpha}{\beta}m}}{\sin(4\pi\frac{\alpha_0}{\beta}m)}\right)^2, & \alpha \notin \{\alpha_{kl}\}, \\ \left(\frac{\mathrm{e}^{2\pi\mathrm{i}\frac{\alpha_0-\alpha}{\beta}m}}{\sin(2\pi\frac{\alpha_0}{\beta}m)}\right)^2, & \alpha \in \{\alpha_{kl}\}, \end{cases} \quad (3.32)$$

and

$$c'_{\alpha,m} = \frac{\check{R}(\alpha)}{\check{G}_\alpha^2\check{R}(0)} \times \begin{cases} \frac{2\cos(4\pi\frac{\alpha_0}{\beta}m)}{\sin^2(4\pi\frac{\alpha_0}{\beta}m)}, & \alpha \notin \{\alpha_{kl}\}, \\ \frac{1}{\sin^2(2\pi\frac{\alpha_0}{\beta}m)}, & \alpha \in \{\alpha_{kl}\}, \end{cases} \quad (3.33)$$

4. Scaling limit in the restricted case: invariant operators, their normalization and vacuum expectation values

Above we have considered unrestricted models. Now let us consider the first restriction for $\delta = 0$. It must contain the trace $\mathrm{Tr}_{\mathcal{H}_{mn}^{(0)}} - \mathrm{Tr}_{\mathcal{H}_{m,-n}^{(0)}}$. Take the combination

$$\tilde{\Pi}_m(a) = \frac{\Pi_m(a) - \Pi_m(-a)}{2\mathrm{i}} = \sum_{n\in\mathbb{Z}+m} \frac{\sin\frac{\pi an}{r}}{\sin\frac{\pi n}{r}}\Pi_{mn}. \quad (4.1)$$

Evidently,

$$\sum_{n\in\mathbb{Z}}[n]\,\mathrm{Tr}_{\mathcal{H}_{mn}^{(0)}}\tilde{\Pi}_m(a)X = \sum_{n=1}^{\infty}[n]\frac{\sin\frac{\pi an}{r}}{\sin\frac{\pi n}{r}}(\mathrm{Tr}_{\mathcal{H}_{mn}^{(0)}}\Pi_{mn}X - \mathrm{Tr}_{\mathcal{H}_{m,-n}^{(0)}}\Pi_{m,-n}X) = \sum_{n=1}^{\infty}[n]\,\mathrm{Tr}_{\mathcal{H}_{mn}^{(1)}}\tilde{\Pi}_m(a)X.$$

It means that for $\delta = 0$ the operators $\tilde{\Pi}_m(a)$ are invariant with respect to the first restriction in Smirnov's sense. Their form factors in the restricted theory RSOS$^{(1)}$ coincide with those in the unrestricted theory.

In the scaling limit the vacuum expectation values of the operator $\tilde{\Pi}_m(a)$ is

$$\tilde{P}_m(a) \simeq \frac{\sin\frac{\pi am}{r-1}}{\sin\frac{\pi m}{r-1}}\left(\frac{M}{4}\right)^{2\Delta(a)} = \frac{\sin(2\pi\frac{\alpha_0-\alpha}{\beta}m)}{\sin(2\pi\frac{\alpha_0}{\beta}m)}\left(\frac{M}{4}\right)^{2\Delta_\alpha}. \quad (4.2)$$

Note that, since for $a \in (r-1)\mathbb{Z}$ the r. h. s. vanishes, the approximation we use fails at these points.

In the scaling limit the operator $\tilde{\Pi}_m(a)$ corresponds to the local field

$$\tilde{\Phi}_\alpha(x) = \frac{\Phi_\alpha(x) - \Phi_{2\alpha_0-\alpha}(x)}{2\mathrm{i}}. \quad (4.3)$$

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Let us find the operators invariant with respect to the second restriction for rational values of r. To provide the subtractions that correspond to the trace over the space $\mathcal{H}_{mn}^{(2)}$ according to the relation (2.15) let us find such values of a that the necessary subtractions take place directly in the trace:

$$\sum_{n\in\mathbb{Z}}[n]\operatorname{Tr}_{\mathcal{H}_{mn}^{(0)}}\tilde{\Pi}_m(a)X = \sum_{n\in\mathbb{Z}}[n]\operatorname{Tr}_{\mathcal{H}_{mn}^{(2)}}\tilde{\Pi}_m(a)X.$$

Indeed, every term in (2.15) can be obtained from the term with $\operatorname{Tr}_{\mathcal{H}_{mn}^{(0)}}$ by a combination of two reflections, $n\to -n$ and $n\to 2q-n$, with simultaneous change of the sign at the trace. As we have seen above, the first reflection property is satisfied for $\tilde{\Pi}_m(a)$ for an arbitrary value of a. The second reflection property is satisfied, if

$$\sin\frac{\pi a(2q-n)}{r} = -\sin\frac{\pi an}{r}.$$

We arrive to the following quantization condition for the parameter a:

$$\frac{aq}{r} = \nu, \qquad \nu\in\mathbb{Z}, \qquad 0 < a < r, \qquad a\notin(r-1)\mathbb{Z}.$$

It means that

$$a = a_\nu \equiv \frac{\nu}{q-p}, \qquad \nu = 1,2,\ldots,q-1, \qquad \nu\notin p\mathbb{Z}. \tag{4.4}$$

Let us represent the value ν in the standard form

$$\nu = \pm(qk-pl), \qquad 0 < k < p, \qquad 0 < l < q. \tag{4.5}$$

The numbers k and l can be restored from the value ν by a simple algorithm (see Appendix). This means that the operator $\tilde{\Pi}_m(a_\nu)$ describes the primary field $\phi_{kl}(x)$ of the conformal dimension Δ_{kl} of the minimal model $M(p,q)$. Evidently,

$$a_\nu = \frac{\alpha_0 - \alpha_{kl}}{\alpha_0}. \tag{4.6}$$

Then

$$\tilde{\Phi}_{kl}(x) \equiv \tilde{\Phi}_{\alpha_{kl}}(x) = N_{kl}\phi_{kl}(x) \tag{4.7}$$

with some normalization factor N_{kl}. Of course, not every primary field $\phi_{kl}(x)$ can be identified with one of the lattice operators $\tilde{\Pi}(a_\nu)$ due to the limitation $0 < \nu < q$. To extract the whole set of primary fields $\phi_{kl}(x)$ in every minimal model $M(p,q)$ we probably needed to study subleading terms in the scaling series for $P_m(a)$.

There are two interesting cases, where the relation between the numbers k, l and the parameter ν is simple. The first one is the unitary series $M(p,p+1)$, where the local height operators describe the fields on the diagonal of the Kac table:

$$\tilde{\Phi}_{kk}(0) = \tilde{\Pi}_m(a_k), \qquad k,m = 1,2,\ldots,p-1.$$

The second case is the series $M(2,2N+1)$, which contains the Lee–Yang model as a particular example $(N=2)$. For this particular series the whole set of primary fields is reproduced:

$$\tilde{\Phi}_{1l} = \tilde{\Pi}_1(a_{2(N-l)+1}), \qquad l = 1,2,\ldots,N.$$

The normalization coefficients N_{kl} in (4.7) can be established using the relations (3.32), (3.33). Consider the product

$$\langle\tilde{\Phi}_\alpha(x)\tilde{\Phi}_\alpha(y)\rangle = |x-y|^{-4\Delta_\alpha}\tilde{c}_{\alpha,m} + \ldots.$$

It is easy to check that

$$\tilde{c}_{\alpha,m} = \frac{\check{R}(\alpha)}{\check{G}_\alpha^2\check{R}(0)}\left(\frac{\sin(2\pi\frac{\alpha_0-\alpha}{\beta}m)}{\sin(2\pi\frac{\alpha_0}{\beta}m)}\right)^2 \quad \text{for } \alpha\in\{\alpha_{kl}\}.$$

Let us normalize the operators ϕ_{kl} as follows:

$$\langle\phi_{kl}(x)\phi_{kl}(y)\rangle = |x-y|^{-4\Delta_{kl}} + \ldots. \tag{4.8}$$

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Then we obtain the normalization constant

$$N_{kl} = \frac{1}{\check{G}_{\alpha_{kl}}} \left(\frac{\check{R}(\alpha_{kl})}{\check{R}(0)} \right)^{1/2} |S_{kl,m}|, \qquad S_{kl,m} = \frac{\sin(2\pi \frac{|\alpha_0 - \alpha_{kl}|}{\beta} m)}{\sin(2\pi \frac{\alpha_0}{\beta} m)}. \tag{4.9}$$

This makes it possible to get the vacuum expectation values of the operators $\phi_{kl}(x)$ using given vacuum expectation values for the sine-Gordon model (3.23). Let

$$\langle \phi_{kl}(x) \rangle_m = H_{kl,m} = (M/4)^{2\Delta_{kl}} \check{H}_{kl,m}. \tag{4.10}$$

By using the expectation value $\tilde{P}_m(a)$ from (4.2) we obtain

$$\check{H}_{kl,m} = Z_{kl,m} \check{H}_{\alpha_{kl}}. \tag{4.11}$$

Here $Z_{kl,m} = \pm 1$ is a sign factor and

$$\check{H}_\alpha = \check{G}_\alpha \left(\frac{\check{R}(0)}{\check{R}(\alpha)} \right)^{1/2} = \left(2\sqrt{\pi} \frac{\Gamma((r+2)/2)}{\Gamma((r-1)/2)} \right)^{2\Delta_\alpha} Q\left(\frac{\alpha - \alpha_0}{\alpha_0} \right), \tag{4.12}$$

$$Q(\eta) = \exp \int_0^\infty \frac{dt}{t} \left(\frac{\operatorname{ch} 2t \operatorname{sh}(\eta-1)t \operatorname{sh}(\eta+1)t}{2 \operatorname{ch} t \operatorname{sh}(r-1)t \operatorname{sh} rt} - (\eta^2 - 1)\alpha_0^2 e^{-4t} \right) \tag{4.13}$$

is the normalization factor obtained in [11]. In fact, we are unable to fix the sign factor $Z_{kl,m}$ since it depends on the definition of signs in the structure constants. It is known to be $(-1)^{(l-1)m}$ for the particular case of $k = 1$ for the choice of structure constants defined in [19]. This result was obtained in different ways [12, 20, 21] and well checked by numerical computations [22, 23]. In the general case it can be conjectured that

$$Z_{kl,m} = \operatorname{sign} S_{kl,m}, \tag{4.14}$$

which is consistent with the abovementioned particular result. It is important to note that for generic k, l the result (4.10), (4.11) contradicts to the conjecture of [12], where the factor $S_{kl,m}$ itself appears in the place of the sign factor $Z_{kl,m}$ in (4.11).

5. Form factors of scaling operators

To proceed with the form factors we need an explicit realization of the corner Hamiltonian and the vertex operators [6, 7]. Consider a set of the oscillators a_k ($k \in \mathbb{Z}$, $k \neq 0$) and a pair of zero mode operators $\mathcal{P}$ and $\mathcal{Q}$ with the commutation relations

$$[\mathcal{P}, \mathcal{Q}] = -i, \qquad [a_k, a_l] = k \frac{\operatorname{sh} \epsilon k \operatorname{sh} \epsilon rk}{\operatorname{sh} 2\epsilon k \operatorname{sh} \epsilon(r-1)k} \delta_{k+l,0}.$$

Define the vacuum states $|m, n\rangle$ ($m, n \in \mathbb{Z} + \delta$) by the relations

$$\mathcal{P}|m, n\rangle = P_{mn}|m, n\rangle, \qquad a_k|m, n\rangle = 0 \quad (k > 0), \qquad P_{mn} = \frac{1}{2}(\alpha_+ m + \alpha_- n).$$

It makes it possible to define the Fock modules $\mathcal{F}_{mn}$ as the span of the vectors of the form

$$a_{-k_1} \ldots a_{-k_N}|m, n\rangle \quad (k_i > 0).$$

There are evidences that, at least for generic δ, the space $\mathcal{F}_{mn}$ can be identified with $\mathcal{H}_{mn}$, so that the trace in (2.20) can be considered as the trace over the space $\mathcal{F}_{mn}$.

Introduce the operator

$$H = \frac{2\epsilon}{\pi} \left(\frac{\mathcal{P}^2 - \alpha_0^2}{2} + \sum_{k=1}^\infty \frac{\operatorname{sh} 2\epsilon k \operatorname{sh} \epsilon(r-1)k}{\operatorname{sh} \epsilon k \operatorname{sh} \epsilon rk} a_{-k} a_k \right).$$

It is identified with the corner Hamiltonian H in (2.20). It is easy to check that $\operatorname{Tr}_{\mathcal{F}_{mn}}(z^{\pi H/2\epsilon}) = \chi_{mn}^{(0)}(z)$.

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Introduce the linear combinations

$$\phi(z) = \frac{\alpha_+}{2}(\mathcal{Q} - i\mathcal{P}\log z) + \sum_{k \neq 0} \frac{a_k}{ik}\, z^{-k}.$$

Introduce two operators

$$V(\theta) = z^{r/4(r-1)} {:}\mathrm{e}^{i\phi(z)}{:}, \qquad \bar{V}(\theta) = z^{r/(r-1)} {:}\mathrm{e}^{-i\phi(x^{-1}z) - i\phi(xz)}{:}.$$

with

$$z = e^{-2\epsilon i\theta/\pi}, \qquad x = e^{-\epsilon}.$$

The normal ordering :...: places a_k with $k > 0$ to the right of those with $k < 0$ and $\mathcal{P}$ to the right of $\mathcal{Q}$. The operator $\bar{V}(\theta)$ is used to introduce the screening operator

$$X_m(\theta) = \int_C \frac{d\gamma}{2\pi}\, \bar{V}(\gamma) F_m(\gamma - \theta), \qquad F_m(\gamma) = \frac{[\gamma/i\pi + 1/2 - m]'}{[\gamma/i\pi - 1/2]'}.$$

The contour C goes over the period of the integrand from $\theta - \pi^2/2\epsilon$ to $\theta + \pi^2/2\epsilon$ along the real axis, but with an inflection: it goes above the point $\theta + i\pi/2$ and below the point $\theta - i\pi/2$.

The vertex operators are given by

$$\Psi^*(\theta)^{m+1}_m = V(\theta),$$
$$\Psi^*(\theta)^{m-1}_m = \eta^{-1} V(\theta) X_m(\theta).$$

Here the constant η is chosen to satisfy the condition (2.18).

Our aim is to obtain the form factors of the operator $\Phi_\alpha(x)$, which are given by the scaling limit of the form factors

$$F_a(\theta_1, \ldots, \theta_N)_{mm_1 \ldots m_{N-1}} \equiv F(\Pi_m(a)|\theta_1, \ldots, \theta_N)_{mm_1 \ldots m_{N-1}}.$$

These form factors contain the sum over n of the form factors (2.20) with some coefficients. The n-dependence of the trace in (2.20) is only contained in the zero mode contribution to the integrand. It can be written as

$$\mathrm{e}^{-2\epsilon P_{mn}^2} \prod_{i \in I_+} \mathrm{e}^{-i\epsilon\alpha_+ P_{m_i n}\theta_i/\pi} \prod_{i \in I_-} \mathrm{e}^{i\epsilon\alpha_+ (2P_{m_i n}\gamma_i - P_{m_i - 2, n}\theta_i)/\pi},$$

where $I_\pm = \{i | m_{i+1} = m_i \pm 1\}$, $m_0 = m_N = m$. In the limit $\epsilon \to 0$ all $P_{m_i n}$ can be substituted by P_{mn} and it reduces to

$$\exp\left(-2\epsilon P_{mn}^2 - i\epsilon\alpha_+ P_{mn}\Theta/\pi\right),$$

where

$$\Theta = \sum_{j=1}^N \theta_j - 2 \sum_{s=1}^{N/2} \gamma_s.$$

Hence, the form factors of $\Pi_m(a)$ are given by

$$F_a(\theta_1, \ldots, \theta_N)_{mm_1 \ldots, m_{N-1}} \simeq \int \prod_{i \in I_-} \left(\frac{d\gamma_i}{2\pi} F_{m_i}(\gamma_i - \theta_i)\right) \left[\frac{\mathrm{Tr}(\mathrm{e}^{-2\pi H} V_N \ldots V_1)}{\eta^{N/2}\,\mathrm{Tr}(\mathrm{e}^{-2\pi H})}\right]_*$$

$$\times \sum_{n \in \mathbb{Z}+m} \frac{[n]\chi_{mn}^{(0)}}{[m]'\chi_m^{(0)}} \frac{\mathrm{e}^{i\pi\frac{an}{r}}}{\sin\frac{\pi n}{r}} \mathrm{e}^{-i\epsilon\alpha_+ P_{mn}\Theta/\pi},$$

where

$$V_i = \begin{cases} V(\theta_i), & i \in I_+ \\ V(\theta_i)\bar{V}(\gamma_i), & i \in I_- \end{cases}$$

and $[\ldots]_*$ means that in all operators and traces we only take into account the contribution of nonzero modes a_k. Let us simplify this expression. Note, that the expression in the brackets has a finite limit for $\epsilon \to 0$. The functions $F_m(\gamma)$ also have a finite limit:

$$F_m(\gamma) \simeq \frac{\mathrm{sh}\,\frac{\gamma + i\pi/2 - i\pi m}{r-1}}{\mathrm{sh}\,\frac{\gamma - i\pi/2}{r-1}}.$$

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At last, the sum over n reduces to

$$\exp\left(-im\frac{\epsilon}{\pi}\frac{r}{r-1}\Theta\right)P_m\left(a+\frac{\epsilon r}{\pi^2}\Theta\right) \simeq \frac{\mathrm{e}^{\mathrm{i}\pi\frac{am}{r-1}}}{\sin\frac{\pi m}{r-1}}\left(\frac{M}{4}\right)^{2\Delta(a)}\mathrm{e}^{-a\Theta/2(r-1)}$$

Finally, in the limit $\epsilon \to 0$ we obtain in

$$F_a(\theta_1,\ldots,\theta_N)_{mm_1\ldots,m_{N-1}}$$
$$= P_m(a)\int \prod_{i\in I_-}\left(\frac{d\gamma_i}{2\pi}F_{m_i}(\gamma_i-\theta_i)\right)\left[\frac{\mathrm{Tr}(\mathrm{e}^{-2\pi H}V_N\ldots V_1)}{\eta^{N/2}\,\mathrm{Tr}(\mathrm{e}^{-2\pi H})}\right]_* \mathrm{e}^{-a\Theta/2(r-1)}. \quad (5.1)$$

It is convenient to rewrite the final answer in the form similar to (2.20). Introduce a continuous set of oscillators $a(t)$ $(t\in\mathbb{R})$ with the commutation relations [24]

$$[a(t),a(t')]=t\frac{\mathrm{sh}\,\frac{\pi t}{2}\,\mathrm{sh}\,\frac{\pi rt}{2}}{\mathrm{sh}\,\pi t\,\mathrm{sh}\,\frac{\pi(r-1)t}{2}}\delta(t+t').$$

Let $|0\rangle$ is the vacuum state defined as

$$a(t)|0\rangle=0,\qquad t>0.$$

It defines a Fock module $\mathcal{F}$ generated by all $a(-t)$ $(t>0)$.

Below we introduce the operators $\phi(\theta)$, $V(\theta)$, $\bar{V}(\theta)$ in terms of $a(t)$ and the constant η, which have not to be confused similar notations above in terms of a_k. Though they are intended to describe the traces in (5.1) in the scaling limit, they differ from the lattice objects.

Let

$$\phi(\theta)=\int_{-\infty}^\infty \frac{dt}{t}\,a(t)\mathrm{e}^{\mathrm{i}\theta t}.$$

Traces of vertex operators contain integrals of the form $\int_0^\infty dt\,f(t)$ with $f(t)$ possessing a pole at the point $t=0$. According to the regularization procedure of [24] they must be interpreted as

$$\int_0^\infty dt\,f(t)=\int_{C_0}dt\,f(t)\frac{\log(-t)}{2\pi\mathrm{i}},$$

where the contour C_0 goes from $+\infty$ above the cut $[0,+\infty)$, then around 0, and then to $+\infty$ below the cut.

Introduce the elementary exponentials

$$V(\theta)=\,:\mathrm{e}^{\mathrm{i}\phi(\theta)}:\,,$$
$$\bar{V}(\theta)=\,:\mathrm{e}^{-\mathrm{i}\phi(\theta+\mathrm{i}\pi/2)-\mathrm{i}\phi(\theta-\mathrm{i}\pi/2)}:\,,$$

where the normal ordering puts $a(t)$ with $t>0$ to the right of $a(-t)$. The screening operator reads

$$X_m(a;\theta)=\int_{C'}\frac{d\gamma}{2\pi}\bar{V}(\gamma)\mathrm{e}^{a\gamma/(r-1)}\frac{\mathrm{sh}\,\frac{\gamma-\theta+\mathrm{i}\pi/2-\mathrm{i}\pi m}{r-1}}{\mathrm{sh}\,\frac{\gamma-\theta-\mathrm{i}\pi/2}{r-1}}.$$

The contour C' goes along the real axis above $\theta+\mathrm{i}\pi/2$ and below $\theta-\mathrm{i}\pi/2$.

It amounts to the following prescription for the scaling type II vertex operators:

$$\Psi^*(a;\theta)_m^{m+1}=V(\theta)\mathrm{e}^{-a\theta/2(r-1)},$$
$$\Psi^*(a;\theta)_m^{m-1}=\eta^{-1}V(\theta)X_m(a;\theta)\mathrm{e}^{-a\theta/2(r-1)}.$$

These operators $\mathrm{e}^{a\theta(m'-m)/2(r-1)}\Psi^*(a;\theta)_m^{m'}$ satisfy the relations (2.16), (2.17) with the S matrix (3.3). With the normalization constant

$$\eta^{-1}=\frac{\mathrm{e}^{a_+^2(C_E+\log\pi(r-1))}}{\pi(r-1)^2}\frac{\Gamma\left(\frac{r}{r-1}\right)}{\Gamma\left(-\frac{1}{r-1}\right)}\exp\int_0^\infty\frac{dt}{t}\frac{\mathrm{sh}\,\frac{\pi t}{2}\,\mathrm{sh}\,\frac{\pi rt}{2}}{\mathrm{sh}\,\pi t\,\mathrm{sh}\,\frac{\pi(r-1)t}{2}}\mathrm{e}^{-\pi t},$$

where C_E is the Euler constant, the vertex operators satisfy the condition

$$\Psi^*(a;\theta')^{m'}_{m''}\Psi^*(a;\theta)^{m''}_{m} = -\frac{\mathrm{i}\sin\frac{\pi m'}{r-1}}{\theta'-\theta-\mathrm{i}\pi}\delta_{m'm} + O(1) \quad \text{as } \theta' \to \theta + \mathrm{i}\pi.$$

Introduce the notation

$$\langle\!\langle X \rangle\!\rangle = \frac{\mathrm{Tr}_{\mathcal{F}}\,\mathrm{e}^{-2\pi H}X}{\mathrm{Tr}_{\mathcal{F}}\,\mathrm{e}^{-2\pi H}}.$$

For the operator X of the form $\prod V(\theta_i)\prod \bar{V}(\gamma_j)$ this quantity is well defined for $z < 1$ and easily calculated by means of the Wick theorem.

The form factors are given by

$$F_a(\theta_1,\ldots,\theta_N)_{mm_1\ldots,m_{N-1}} = P_m(a)\langle\!\langle \Psi^*(a;\theta_N)^m_{m_{N-1}}\ldots\Psi^*(a;\theta_1)^{m_1}_m \rangle\!\rangle, \tag{5.2}$$

with $P_m(a)$ given by (3.10). By checking the cyclicity and kinetic residue properties of this form factor we can make sure that the operator $\Phi_\alpha(x)$ is local with respect to the excitation creating operators. It means that all Euclidean correlation functions that only contain the operators $\Phi_\alpha(x)$ at some points x_i and the bosonic excitation creating operators are well-defined single-valued functions of x_i. Hence, the operator $\tilde{\Phi}_\alpha(x)$, which possesses the form factors

$$\tilde{F}_a(\theta_1,\ldots)_{mm_1\ldots} = \frac{1}{2\mathrm{i}}\left(F_a(\theta_1,\ldots)_{mm_1\ldots} - F_{-a}(\theta_1,\ldots)_{mm_1\ldots}\right), \tag{5.3}$$

is also local with respect to excitation creating operators. On the contrary, the operators $\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}$ are essentially nonlocal due to the m dependence of the coefficient $B_{\alpha,m}$ (3.26). These operators are only local in the II$_-$ limit, while in the II$_+$ limit the operators $\mathrm{e}^{\mathrm{i}\alpha\varphi'(x)}$ with $\varphi'(x)$ defined in (3.31) are local.

Let us discuss validity of the formulas (5.2), (5.3). The scaling limit was calculated in the region $-r < a < r$. It coincides with the region of convergence of integrals in γ_i. As analytic functions in a the scaling form factors (5.2) and (5.3) can be continued to the whole complex plane. One may conjecture that this analytic continuation gives the form factors for any exponential field $\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}$ with the relation (3.12) in the unrestricted theory. For the RSOS$^{(1)}$ and RSOS$^{(2)}$ models we may think that the analytic continuation of the form factors for the operators $\tilde{\Pi}_m(a)$ describes the form factors of all primary operators $\phi_{kl}(x)$, which correspond to $\alpha = \alpha_{kl}$. To validate this conjecture it would be necessary either to treat more accurately the subleading contributions to the scaling form factors or to consider some multipoint local height operators. We defer this task to future work.

Let us return to the scaling limit II$_-$. Redefine the Ψ^* operators:

$$\Psi^*_+(\alpha;\theta) = \Psi^*(a;\theta)^{m+1}_m \times \mathrm{e}^{\theta/2(r-1)}, \qquad \Psi^*_-(\alpha;\theta) = \Psi^*(a;\theta)^{m-1}_m \times 2\mathrm{i}\mathrm{e}^{-\theta/2(r-1)}\mathrm{e}^{-\mathrm{i}\pi\frac{m-1}{r-1}}.$$

We get for the form factors

$$F_a(\theta_1,\ldots,\theta_N)_{mm_1\ldots m_{N-1}} \simeq 2\mathrm{i}\left(\frac{M}{4}\right)^{\alpha^2}\prod_{i\in I_+}\mathrm{e}^{\theta_i/2(r-1)}\prod_{i\in I_-}\mathrm{e}^{-\theta_i/2(r-1)}\mathrm{e}^{-\mathrm{i}\pi\frac{m_i-1}{r-1}}$$
$$\times\,\mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s}\langle\!\langle \Psi^*_{\varepsilon_N}(\alpha;\theta_N)\ldots\Psi^*_{\varepsilon_1}(\alpha;\theta_1)\rangle\!\rangle, \qquad \varepsilon_i = m_i - m_{i-1}. \tag{5.4}$$

Explicitly, these operators are given by

$$\Psi^*_+(\alpha;\theta) = V(\theta)\mathrm{e}^{\alpha\theta/\beta},$$
$$\Psi^*_-(\alpha;\theta) = \eta^{-1}V(\theta)X(\alpha;\theta)\mathrm{e}^{\alpha\theta/\beta}$$

with

$$X(\alpha;\theta) = \int_{C'}\frac{d\gamma}{2\pi}\bar{V}(\gamma)\frac{e^{-2\alpha\gamma/\beta}}{\mathrm{sh}\frac{\gamma-\theta+\mathrm{i}\pi/2}{r-1}}.$$

Physically, the second line of (5.4) represents the form factor of the field $\check{G}^{-1}_\alpha\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}$. These form factors are written for the S matrix defined in (3.6). If we want to obtain the form factors for the pure sine-Gordon S matrix $S_{SG;\beta}(\theta)$, we need to make a simple gauge transformation. Finally, we reproduce Lukyanov's result [9] for the exponential fields:

$$F(\mathrm{e}^{\mathrm{i}\alpha\varphi}|\theta_1,\ldots,\theta_N)_{\varepsilon_1\ldots\varepsilon_N,s} = \mathrm{e}^{2\pi\mathrm{i}\frac{\alpha}{\beta}s}G_\alpha\langle\!\langle \Psi^*_{\varepsilon_N}(\alpha;\theta_N)\ldots\Psi^*_{\varepsilon_1}(\alpha;\theta_1)\rangle\!\rangle.$$

The factor $e^{2\pi i\frac{\alpha}{\beta}s}$ makes the operator $e^{i\alpha\varphi(x)}$ be formally local with respect to soliton creating operators. This locality not physical. The definition of the soliton creating operators in the non-compact theory in terms of the field φ inevitably provides cuts, at which all correlation functions are discontinuous. If we try to match the correlation functions at both banks of each cut, we shall obtain a continuous but multivalued functions, which correspond to the known mutual quasilocality of the neutral exponential operators and the soliton creating operators.

6. Discussion

We discussed the scaling limits in the SOS and RSOS models aiming to clarify some problems in the corresponding integrable quantum field theories. We saw that this approach makes it possible to obtain some expressions for form factors and sheds some light on the relation between the unrestricted and restricted sine-Gordon theories. Two scaling limits were considered, one of which gave us the sine-Gordon theory, while the second one admitted the restrictions to the perturbed minimal models. The same operators in the lattice theory can be identified in both limits with local operators in quantum field theory, which allows one to relate their vacuum expectation values.

On the other hand the above consideration fetches out some fundamental problems to be solved in future. First of all, the correct description of the SOS vacuums (or, in other words, the quantum group invariant vacuums) in the sine-Gordon theory is absent. This is closely related to the fact that the vertex-face correspondence, which is studied in much detail for local variables of the lattice models, is poorly studied on the level of excitations.

The second problem is understanding the special fields in the sine-Gordon theory. It is known from practice that many results concerning the *special* fields in the minimal conformal field theory and sine-Gordon theory can be obtained by formal 'analytic continuation' of the corresponding results for *generic* fields in Liouville and sinh-Gordon theories. But there is no firm basis for this procedure. Moreover, it is not clear why the operator product expansions in the Liouville and sinh-Gordon theories give a continuous spectrum of fields, while the corresponding expansions in the minimal CFTs and sine-Gordon theory are discrete. Another aspect of this problem is how the $2/\beta$ shifts of the parameter of the exponential operators appear in the sine-Gordon theory that only contains the $\cos\beta\varphi$ perturbation.

I hope that these notes will stimulate efforts to solve these problems.

Appendix

In this Appendix we prove the

Theorem 1 *Let p and q be coprimes such that $q > p > 0$ and[2]*

$$\nu \in \mathbb{Z}, \qquad |\nu| \neq qa + pb \quad \forall a, b \in \mathbb{Z}_{\geq 0}, \qquad |\nu| < pq. \tag{A.1}$$

Then the equation

$$qk - pl = \nu \tag{A.2}$$

has a unique solution in integers k, l in the domain

$$0 < k < p, \qquad 0 < l < q. \tag{A.3}$$

---

[2]The last condition for ν is excessive being the consequence of the previous one.

In particular, the hypothesis of the Theorem is valid for the equation (4.5) with the conditions (4.4). Uniqueness of a solution is evident. Let us prove existence.

Let $\alpha = (p, q, k, l, \nu) \in \mathbb{Z}^5$. Consider transformations on $\mathbb{Z}^5$ preserving the form of the equation (A.2). Let n be a positive integer. Then we may rewrite the equation (A.2) as

$$p(l - nk) - (q - np)k = -\nu.$$

It makes it possible to define the transformation

$$T_n : \alpha \mapsto \alpha_1, \qquad p_1 = q - np, \qquad q_1 = p, \qquad k_1 = l - nk, \qquad l_1 = k, \qquad \nu_1 = -\nu.$$

The numbers p_1, q_1 are again coprimes. This transformation is invertible in integers, namely

$$T_n^{-1} : \alpha_1 \mapsto \alpha, \qquad p = q_1, \qquad q = p_1 + nq_1, \qquad k = l_1, \qquad l = k_1 + nl_1, \qquad \nu = -\nu_1.$$

After the transformations T_n we have
$$q_1 k_1 - p_1 l_1 = \nu_1.$$

Chose n to satisfy the conditions

$$0 < q - pn < p \qquad \Leftrightarrow \qquad 0 < p_1 < q_1.$$

This condition determines the value of n uniquely for $p > 1$. Subject to this condition we have $p_1 < p$, $q_1 < q$. Now chose n_1 so that $0 < q_1 - p_1 n_1 < p_1$. Let $\alpha_2 = T_{n_1} \alpha_1 = T_{n_1} T_n \alpha$. Iterate the procedure:

$$\alpha_{t+1} = T_{n_t} \alpha_t, \qquad 0 < q_t - p_t n_t < p_t,$$

till $p_N = 1$. The last equation reads

$$q_N k_N - l_N = \nu_N \equiv (-1)^N \nu. \tag{A.4}$$

The numbers n, n_t, q_N can be easily defined by the simple continued fraction

$$\frac{q}{p} = [n_0, n_1, n_2, \ldots, n_{N-1}, n_N], \qquad n_0 = 0, \qquad n_N = q_N > 1, \tag{A.5}$$

where we use the notation

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}.$$

It is easy to solve the equation (A.4) by fixing any value of k_N and taking $l_N = q_N k_N - (-1)^N \nu$. Then we can calculate k, l using the inverse transformations as

$$\alpha = T_n^{-1} T_{n_1}^{-1} \ldots T_{n_N}^{-1} \alpha_N.$$

The question is if the answer belongs to the domain (A.3). To solve this problem let us rewrite the conditions (A.3) in terms of α_N. First, let us write down α in terms of α_t:

$$\begin{aligned} k &= A_t k_t + B_t l_t, & p &= A_t p_t + B_t q_t, \\ l &= C_t k_t + D_t l_t, & q &= C_t p_t + D_t q_t. \end{aligned}$$

The coefficients $A_t, \ldots, D_t$ satisfy the recursion relations

$$\begin{aligned} A_{t+1} &= B_t, & C_{t+1} &= D_t, \\ B_{t+1} &= A_t + n_t B_t, & D_{t+1} &= C_t + n_t D_t \end{aligned} \tag{A.6}$$

with the initial conditions $A_0 = 1$, $B_0 = 0$, $C_0 = 0$, $D_0 = 1$.

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It is convenient to introduce the ratios

$$x_t = \frac{B_t}{A_t} = [n_{t-1}, n_{t-2}, \ldots, n_1] \geq 1,$$

$$y_t = \frac{D_t}{C_t} = [n_{t-1}, n_{t-2}, \ldots, n_1, n_0] \geq 1.$$

Hence,

$$x_t = n_{t-1} + x_{t-1}^{-1},$$
$$y_t = n_{t-1} + y_{t-1}^{-1}. \tag{A.7}$$

The conditions (A.3) can be rewritten as

$$0 < x_N^{-1} k_N + l_N < x_N^{-1} + q_N,$$
$$0 < y_N^{-1} k_N + l_N < y_N^{-1} + q_N.$$

Let rewrite these conditions in terms of k_N and $\nu_N = (-1)^N \nu$:

$$(q_N + x_N^{-1})(k_N - 1) < \nu_N < (q_N + x_N^{-1})k_N,$$
$$(q_N + y_N^{-1})(k_N - 1) < \nu_N < (q_N + y_N^{-1})k_N. \tag{A.8}$$

The regions defined by these two lines have nonempty intersection in $\mathbb{R}$, if

$$(q_N + x_N^{-1})(k_N - 1) < (q_N + y_N^{-1})k_N, \qquad (q_N + y_N^{-1})(k_N - 1) < (q_N + x_N^{-1})k_N.$$

This can be considered as a weak compatibility condition for the inequalities (A.8). It can be rewritten as

$$-q_N - y_N^{-1} < (x_N^{-1} - y_N^{-1})k_N < q_N + x_N^{-1}. \tag{A.9}$$

Calculate the coefficient at k_N. We have

$$x_t^{-1} - y_t^{-1} = (y_t - x_t)x_t^{-1}y_t^{-1} = -(x_{t-1}^{-1} - y_{t-1}^{-1})x_t^{-1}y_t^{-1}, \qquad x_1^{-1} = 0.$$

Therefore,

$$x_N^{-1} - y_N^{-1} = (-1)^N \prod_{t=2}^{N} x_t^{-1} \prod_{t=1}^{N} y_t^{-1} = (-1)^N B_N^{-1} D_N^{-1}.$$

It is easy to check that for both even and odd N the compatibility condition (A.9) in combination with the strictest of the conditions of (A.8) provides the condition

$$|\nu| = |\nu_N| < \frac{(q_N + x_N^{-1})(q_N + y_N^{-1})}{|x_N^{-1} - y_N^{-1}|} = (q_N + x_N^{-1})(q_N + y_N^{-1})B_N D_N. \tag{A.10}$$

It is evident that

$$(q_N + x_N^{-1})B_N = p, \qquad (q_N + y_N^{-1})D_N = q. \tag{A.11}$$

Substituting (A.11) to (A.10) we obtain the last of the conditions (A.1):

$$|\nu| < pq.$$

Consider now the conditions (A.8) more accurately and find the compatibility conditions in integers. We want to find the values of $\nu_N = \xi_{N+1}$ for which these inequalities are NOT satisfied. The respective value of k_N will be denoted as ξ_N (the reason for this notation will be clear later). Let us take, for definiteness, the case $\nu_N \geq 0$. Consider first the case $N \in 2\mathbb{Z}$. In this case we can take

$$(q_N + y_N^{-1})\xi_N \leq \xi_{N+1} \leq (q_N + x_N^{-1})\xi_N. \tag{A.12}$$

With the definition

$$x_{N+1} = q_N + x_N^{-1}, \qquad y_{N+1} = q_N + y_N^{-1}$$

we can write it also as
$$y_{N+1}\xi_N \le \xi_{N+1} \le x_{N+1}\xi_N.$$

We have to solve this equation for ξ_{N+1}, ξ_N in nonnegative integers. Let us take
$$\xi_{N+1} = q_N\xi_N + \xi_{N-1}.$$

Then using (A.7) the inequality (A.12) is rewritten as
$$(n_{N-1} + x_{N-1}^{-1})\xi_{N-1} \le \xi_N \le (n_{N-1} + y_{N-1}^{-1})\xi_{N-1}$$

or
$$x_N\xi_{N-1} \le \xi_N \le y_N\xi_{N-1}.$$

Iterating this procedure we get
$$\xi_t = n_{t-1}\xi_{t-1} + \xi_{t-2} \tag{A.13}$$

with the conditions
$$y_t\xi_{t-1} \le \xi_t \le x_t\xi_{t-1} \quad \text{for } t \in 2\mathbb{Z}+1,$$
$$x_t\xi_{t-1} \le \xi_t \le y_t\xi_{t-1} \quad \text{for } t \in 2\mathbb{Z}.$$

Note that for all values of t these conditions are equivalent. Hence, starting from $N = 2\mathbb{Z}+1$ we can derive the same inequalities. From now on we do not restrict the consideration to the case of even N.

Taking into account that $x_0 = 0$, $y_0 = +\infty$ we arrive to the conditions
$$\xi_{-1}, \xi_0 \ge 0. \tag{A.14}$$

Express all ξ_t in terms of ξ_0, ξ_{-1}. It is easy to prove that
$$\xi_t = B_t\xi_{-1} + D_t\xi_0. \tag{A.15}$$

Indeed, $\xi_1 = \xi_{-1} + n\xi_0$ according to (A.13), which conforms (A.15). Besides, from (A.13) we get the recursion relation
$$B_{t+1} = n_t B_t + B_{t-1}, \qquad D_{t+1} = n_t D_t + D_{t-1},$$

which conforms (A.6).

Using (A.5) and (A.11) we get
$$B_{N+1} = A_N + q_N B_N = (q_N + x_N^{-1})B_N = p, \qquad D_{N+1} = C_N + q_N D_N = (q_N + y_N^{-1})D_N = q.$$

Finally we get that the values
$$\xi_{N+1} = p\xi_{-1} + q\xi_0 \qquad (\xi_{-1}, \xi_0 \ge 0)$$

are the only forbidden values of $|\nu_N| = |\nu|$, which proves the theorem.

The constructive way to find k and l can be extracted from this proof and formulated as

Theorem 2 *Under the hypothesis of the Theorem 1 define the numbers $n_0, \ldots, n_N$ by the simple continuous fraction*
$$\frac{q}{p} = [n_0, n_1, \ldots, n_N], \qquad n_N > 1.$$

Consider any solution k_N, l_N to the equation
$$n_N k_N - l_N = \nu_N \equiv (-1)^N \nu.$$

Consider the recurrent relations
$$k_{t-1} = l_t, \qquad l_{t-1} = k_t + n_{t-1}l_t.$$

Let $k' = k_0$, $l' = l_0$. Then there exists such $s \in \mathbb{Z}$ that $k = k' - sp$, $l = l' - sq$ is the solution to the equation (A.2) satisfying the condition (A.3).

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The proof is simple. Consider the transformation

$$S : \alpha \mapsto \alpha', \qquad p' = p, \qquad q' = q, \qquad k' = k - p, \qquad l' = l - q, \qquad \nu' = \nu.$$

It is easy to check that it commutes with T_n:

$$T_n S = S T_n.$$

We have proved that there exists a solution k_N, l_N to the equation (A.4) corresponding to the solution $k = k_0$, $l = l_0$ to the equation (A.2) such that $0 < k < p$, $0 < l < q$. Evidently, for any given solution k'_N, l'_N to the equation (A.4) there exists $s \in \mathbb{Z}$ such that $(1, q_N, k_N, l_N, \nu_N) = S^s(1, q_N, k'_N, l'_N, \nu_N)$. Then $(p, q, k, l, \nu) = S^s(p, q, k', l', \nu)$.

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